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The Electric or Magnetic Polarisation of a Thin Cylinder of Finite Length by a Uniform Field of Force.

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§ 1. *Introduction.*

The problem to be considered is the calculation of the polarisation induced in a cylindrical rod of finite length by a uniform field of force parallel to its axis.

The only formula previously given appears to be that of Green* for the normal component of the polarity at the surface of the rod. In the following investigation the method of solution of the general problem of induction by series is used, and expressions are given in §§ 4 and 5 for the mean moment of the rod and the induction across the central section. It is assumed that the susceptibility of the rod is independent of the applied force; consequently the formulæ apply especially to a dielectric rod suspended in a uniform field of electric force, this being, in fact, the purpose for which the calculations were made.

In the case ong thin rods in a magnetic field it has been assumed that the demagnetising factor might be taken as that of an ellipsoid inscribable in the rod; but this neglects the fact of the non-uniformity of the field within the rod, and various experimental investigations have shown that the simple hypothesis is untenable. In § 7 these experimental results are discussed in relation to the formulæ obtained in the previous sections.

§ 2. *The Series Solution of the General Problem.*

Consider a body placed in a magnetic field whose potential is V, and let a constant κ denote the magnetic susceptibility of the substance. Let Q be the potential of the induced magnetism and ϕ the total potential; then we have

$$\phi = Q + V.$$

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* Green, 'Mathematical Papers,' p. 106; also Maxwell, 'Electricity and Magnetism' vol. 2, § 439.

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The components of the intensity of magnetisation I at any point are given by

$$\mathrm{I} = -\kappa\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\phi.$$

If $d\nu$ be the inward-drawn normal at the surface of the body, the problem to be solved in finding Q may be expressed in the equation

$$\mathrm{Q} = \kappa\int\frac{\partial(\mathrm{V}+\mathrm{Q})}{\partial\nu}\frac{d\sigma}{r}, \tag{1}$$

where the integral extends over the surface of the body.

The exact solution of this problem can only be found in a limited number of cases, as, for example, when the surface of the body is an ellipsoid. A method of approximating to a solution was given by Beer,[*] and has been further developed by Neumann,[†] Riecke,[‡] and Wassmuth.[§] The method may be regarded physically as one of successive superpositions in the manner of Murphy.[||] Suppose, as a first approximation, that the potential V produces an induction I_1 equal to $-\kappa(\partial/\partial x, \partial/\partial y, \partial/\partial z)\mathrm{V}$; then the distribution I_1 will have a potential V_1, which may in its turn be supposed to induce a distribution I_2, and so on. The total induction will be given by the sum of the series I; and the success of the method depends upon the convergency of this series.

Defining a series of functions by the relations

$$\mathrm{V}_1 = \int\frac{\partial\mathrm{V}}{\partial\nu}\frac{ds}{r}; \qquad \mathrm{V}_2 = \int\frac{\partial\mathrm{V}_1}{\partial\nu}\frac{ds}{r}; \text{ and so on,} \tag{2}$$

the first form of series for the induced potential Q, given by Beer, was

$$\mathrm{Q} = 4\pi\kappa\mathrm{V}_1+(4\pi\kappa)^2\mathrm{V}_2+(4\pi\kappa)^3\mathrm{V}_3+\dots . \tag{3}$$

But even if the intermediary potentials decrease in magnitude, as in most practical cases, this series for Q is convergent only for a limited range of values for κ. Other forms of series have been obtained which include Beer's series, but give a wider range of convergency; two such series given by Riecke and Neumann are equivalent and are expressed in terms of the functions $\mathrm{V}_1, \mathrm{V}_2 \dots$ by

$$\mathrm{Q} = \rho\mathrm{V}_1+\rho^2(\mathrm{V}_1+\mathrm{V}_2)+\rho^3(\mathrm{V}_1+2\mathrm{V}_2+\mathrm{V}_3)+\dots , \tag{4}$$

where $\rho = 4\pi\kappa/(1+4\pi\kappa) = 4\pi\kappa/\mu$.

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[*] Beer, 'Elektricität und Magnetismus,' p. 155.

[†] C. Neumann, 'Das logarithm. Potential,' p. 248.

[‡] Riecke, 'Wied. Ann.,' vol. 13, p. 466, 1881.

[§] Wassmuth, 'Wied. Ann.,' vol. 51, p. 367, 1894.

[||] Murphy, 'Electricity,' p. 93, 1833.

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This series is certainly convergent for all practical cases in which the intermediary potentials form a decreasing series, for instance, in the case of an ellipsoid placed in a uniform field parallel to an axis;* for then we find

$$V = -Hx; \qquad V_1 = \frac{L}{4\pi}Hx; \qquad V_2 = -\left(\frac{L}{4\pi}\right)^2 Hx; \; \ldots\ldots;$$

where L is the usual constant associated with an ellipsoid and is less than 4π.

Forming the functions for series (4) we obtain

$$\rho V_1 = \frac{\kappa}{\mu}LHx; \qquad \rho^2(V_1+V_2) = \frac{(4\pi-L)\kappa}{\mu}\,\frac{\kappa}{\mu}LHx;$$

$$\rho^3(V_1+2V_2+V_3) = \frac{(4\pi-L)^2\kappa^2}{\mu^2}\,\frac{\kappa}{\mu}LHx, \quad \text{etc.}$$

Hence we have

$$Q = \frac{\kappa}{\mu}LHx\left\{1+\frac{(4\pi-L)\kappa}{\mu}+\frac{(4\pi-L)^2\kappa^2}{\mu^2}+\ldots\right\}.$$

But this series is convergent for all values of κ, and in fact gives the known exact solution

$$Q = \frac{\kappa L}{1+\kappa L}Hx.$$

We shall use, then, the series (4) in the following work, in which we consider the case of a long rod in a uniform field.

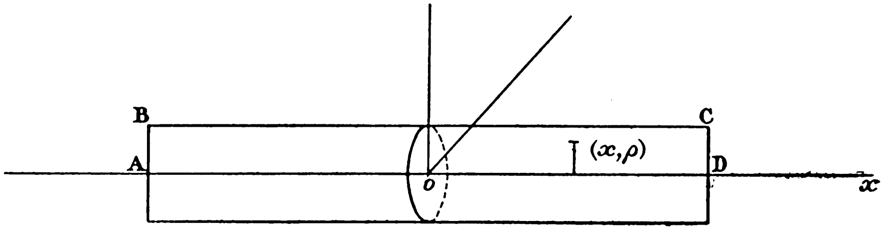
§ 3. *Series for a Finite Rod.*

We shall consider the case of a cylindrical rod of length $2l$, with flat circular ends of radius a; we write $l/a = m$.

Suppose the rod placed in a uniform field H parallel to its axis, which we take as the axis of x; then we have

$$V = -Hx. \tag{5}$$

Our object is to calculate the functions $V_1, V_2, \ldots$ defined in (2).



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* *Cf.* Riecke, *loc. cit.*, p. 481.

Using cylindrical co-ordinates (x, ρ, θ), we have in general

$$4\pi V_{n+1}(x, \rho) = \int \frac{\partial V_n(x', \rho')}{\partial \nu} \frac{d\sigma}{r} = \int \frac{\partial V_n}{\partial \nu} \frac{d\sigma}{\sqrt{(x'-x)^2+\rho^2+\rho'^2-2\rho\rho' \cos\theta'}}. \quad (6)$$

Also the element of surface $d\sigma$ is equal to $\rho' ds\, d\theta'$, where ds is equal to dx' or $d\rho'$ according as the point (x', ρ') is on the curved side or a flat end of the rod; and putting

$$R^2 = \rho^2+\rho'^2-2\rho\rho' \cos\theta',$$

we have

$$4\pi V_{n+1}(x, \rho) = \int_0^{2\pi} d\theta' \int \rho' \frac{\partial V_n}{\partial \nu} ds \int_0^\infty e^{\pm\lambda(x'-x)} J_0(\lambda R)\, d\lambda$$

$$= 2\pi \int \rho' \frac{\partial V_n}{\partial \nu} ds \int_0^\infty e^{\pm\lambda(x'-x)} J_0(\lambda\rho)\, J_0(\lambda\rho')\, d\lambda, \quad (7)$$

where the integral with respect to ds extends over the path ABCD, and the upper or lower sign is taken in the exponential according as $x'-x$ is negative or positive.

Thus we find

$$\begin{aligned} V_{n+1}(x, \rho) = {} & \tfrac{1}{2} \int_0^a \left(\frac{\partial V_n}{\partial x'}\right)_{x'=-l} \rho' d\rho' \int_0^\infty e^{-\lambda(x+l)} J_0(\lambda\rho)\, J_0(\lambda\rho')\, d\lambda \\ & -\tfrac{1}{2} \int_0^a \left(\frac{\partial V_n}{\partial x'}\right)_{x'=l} \rho' d\rho' \int_0^\infty e^{-\lambda(l-x)} J_0(\lambda\rho)\, J_0(\lambda\rho')\, d\lambda \\ & -\tfrac{1}{2} \int_{-l}^x \left(\frac{\partial V_n}{\partial \rho'}\right)_{\rho'=a} a\,dx' \int_0^\infty e^{\lambda(x'-x)} J_0(\lambda\rho)\, J_0(\lambda a)\, d\lambda \\ & -\tfrac{1}{2} \int_x^l \left(\frac{\partial V_n}{\partial \rho'}\right)_{\rho'=a} a\,dx' \int_0^\infty e^{-\lambda(x'-x)} J_0(\lambda\rho)\, J_0(\lambda a)\, d\lambda. \quad (8) \end{aligned}$$

From symmetry in the case under consideration, this reduces to

$$\begin{aligned} V_{n+1}(x, \rho) = {} & -\int_0^a \left(\frac{\partial V_n}{\partial x'}\right)_{x'=l} \rho' d\rho' \int_0^\infty e^{-\lambda l} \sinh(\lambda x)\, J_0(\lambda\rho)\, J_0(\lambda\rho')\, d\lambda \\ & -\int_x^l \left(\frac{\partial V_n}{\partial \rho'}\right)_{\rho'=a} a\,dx' \int_0^\infty e^{-\lambda x'} \sinh(\lambda x)\, J_0(\lambda\rho)\, J_0(\lambda a)\, d\lambda \\ & -\int_0^x \left(\frac{\partial V_n}{\partial \rho'}\right)_{\rho'=a} a\,dx' \int_0^\infty e^{-\lambda x} \cosh(\lambda x')\, J_0(\lambda\rho)\, J_0(\lambda a)\, d\lambda. \quad (9) \end{aligned}$$

Thus at any stage in the evaluation of the successive potentials, V_{n+1} is expressed as the potential of a surface distribution due to V_n; the first integral in (9) gives the part due to the flat ends of the rod and the other two integrals that due to the curved sides.

The potential V_1 can be deduced from (9), or obtained directly as the potential of two uniformly charged circular discs situated at the ends of the rod.

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Using the relation

$$\int_0^a \rho' \, \mathrm{J}_0(\lambda\rho') \, d\rho' = a\mathrm{J}_1(\lambda a)/\lambda,$$

we obtain

$$\mathrm{V}_1 = \mathrm{H}a \int_0^\infty e^{-\lambda l} \sinh(\lambda x) \, \mathrm{J}_0(\lambda\rho) \, \mathrm{J}_1(\lambda a) \, d\lambda/\lambda. \tag{10}$$

§ 4. *Calculation of the Total Moment of the Rod.*

The magnetisation at any point of the rod is equal to $-\kappa\,(\partial/\partial x, \partial/\partial y, \partial/\partial z)\,\phi$, where ϕ is the total potential, and the moment of the rod can be calculated by considering the normal component of the polarisation over the surface of the rod. Writing M for the total moment, we have

$$\mathrm{M} = -2l \int_0^a 2\pi\kappa\rho \left(\frac{\partial\phi}{\partial x}\right)_{x=l} d\rho - 2\int_0^l 2\pi\kappa a x \left(\frac{\partial\phi}{\partial\rho}\right)_{\rho=a} dx, \tag{11}$$

where the first integral gives the part due to the ends of the rod, and the second integral that due to the sides.

Now ϕ is given in terms of V and the intermediary potentials V_n according to the series given in (3) and (4); thus from (11) we find that it is required to evaluate expressions of the form

$$\mathrm{U}_n = 2l \int_0^a 2\pi\rho \left(\frac{\partial \mathrm{V}_n}{\partial x}\right)_{x=l} d\rho + 2\int_0^l 2\pi a x \left(\frac{\partial \mathrm{V}_n}{d\rho}\right)_{\rho=a} dx. \tag{12}$$

We shall consider these expressions as far as U_2, and obtain the important terms in their expansions in powers of $1/m$, where m is the ratio of the length of the rod to its diameter, and is assumed to be large.

Then since

$$\mathrm{V} = -\mathrm{H}x,$$

we have

$$\mathrm{U} = -2\pi a^2 l\kappa\mathrm{H}. \tag{13}$$

Further, from the value of V_1 given in (10) we have

$$\mathrm{U}_1 = 4\pi a^2 l\mathrm{H} \int_0^\infty e^{-\lambda l}\lambda^{-1} \cosh(\lambda l) \, \mathrm{J}_1{}^2(\lambda a) \, d\lambda$$
$$-4\pi a^2\mathrm{H} \int_0^l x\,dx \int_0^\infty e^{-\lambda l} \sinh(\lambda x) \, \mathrm{J}_1{}^2(\lambda a) \, d\lambda. \tag{14}$$

If in the second part we perform the integration with respect to x, and put μ for λa, we find that (14) reduces to

$$\mathrm{U}_1 = 2\pi a^3\mathrm{H} \int_0^\infty \frac{\mathrm{J}_1{}^2(\mu)}{\mu^2} d\mu - 2\pi a^3\mathrm{H} \int_0^\infty e^{-2m\mu} \frac{\mathrm{J}_1{}^2(\mu)}{\mu^2} d\mu. \tag{15}$$

Now we have

$$\int_0^\infty \frac{\mathrm{J}_1{}^2(\mu)}{\mu^2} d\mu = \frac{4}{3\pi}.$$

Also, since m is large, the important terms of the second integral are obtained by substituting the value of the Bessel function for small values of the argument; thus as far as the term m^{-2} we have

$$\int_0^\infty e^{-2m\mu} \frac{J_1^2(\mu)}{\mu^2}\, d\mu = \frac{1}{8m}. \tag{16}$$

Hence to the same order we have

$$\begin{aligned} U_1 &= 2\pi a^3 H \left(\frac{4}{3\pi} - \frac{1}{8m}\right) \\ &= 2\pi a^2 l H \left(\frac{4}{3\pi m} - \frac{1}{8m^2}\right). \end{aligned} \tag{17}$$

Now we have to evaluate U_2 in a similar manner; we have from the general expression given in (9),

$$\begin{aligned} \left(\frac{\partial V_2}{\partial x}\right)_{x=l} &= -\int_0^a \left(\frac{\partial V_1}{\partial x'}\right)_l \rho' d\rho' \int_0^\infty \lambda e^{-\lambda l} \cosh(\lambda l)\, J_0(\lambda\rho)\, J_0(\lambda\rho')\, d\lambda \\ &\quad + \int_0^l \left(\frac{\partial V_1}{\partial \rho'}\right)_a a dx' \int_0^\infty \lambda e^{-\lambda l} \cosh(\lambda x')\, J_0(\lambda\rho)\, J_0(\lambda a)\, d\lambda \\ &= -\tfrac{1}{2}\left(\frac{\partial V_1}{\partial x}\right)_l - \tfrac{1}{2} \int_0^a \left(\frac{\partial V_1}{dx'}\right)_l \rho' d\rho' \int_0^\infty \lambda e^{-2\lambda l} J_0(\lambda\rho)\, J_0(\lambda\rho')\, d\lambda \\ &\quad + \int_0^l \left(\frac{\partial V_1}{\partial \rho'}\right)_a a dx' \int_0^\infty \lambda e^{-\lambda l} \cosh(\lambda x')\, J_0(\lambda\rho)\, J_0(\lambda a)\, d\lambda. \end{aligned} \tag{18}$$

And, similarly, from (9) we have

$$\left(\frac{\partial V_2}{\partial \rho}\right)_{=a} = -\tfrac{1}{2}\left(\frac{\partial V_1}{\partial \rho}\right)_a + \int_0^a \left(\frac{\partial V_1}{\partial x'}\right)_l \rho' d\rho' \int_0^\infty \lambda e^{-\lambda l} \sinh(\lambda x)\, J_1(\lambda a)\, J_0(\lambda\rho')\, d\lambda. \tag{19}$$

It may be noted that if the intermediary potentials V_n be regarded as due to successive surface distributions σ_n, then the equations (18) and (19) are examples of general relations between consecutive surface densities.

Using the values of (18) and (19), we have

$$\begin{aligned} U_2 = &-l \int_0^a 2\pi\rho \left(\frac{\partial V_1}{\partial x}\right)_l d\rho - l \int_0^a 2\pi\rho d\rho \int_0^a \left(\frac{\partial V_1}{\partial x'}\right)_l \rho' d\rho' \int_0^\infty \lambda e^{-2\lambda l} J_0(\lambda\rho)\, J_0(\lambda\rho')\, d\lambda \\ &+ 2l \int_0^a 2\pi\rho d\rho \int_0^l \left(\frac{\partial V_1}{\partial \rho'}\right)_a a dx' \int_0^\infty \lambda e^{-\lambda l} \cosh(\lambda x')\, J_0(\lambda\rho)\, J_0(\lambda a)\, d\lambda \\ &- \int_0^l 2\pi a x \left(\frac{\partial V_1}{\partial \rho}\right)_a dx + 2 \int_0^l 2\pi a x dx \int_0^a \left(\frac{\partial V_1}{\partial x'}\right)_l \rho' d\rho' \int_0^\infty \lambda e^{-\lambda l} \sinh(\lambda x)\, J_1(\lambda a) \\ &\qquad J_0(\lambda\rho')\, d\lambda. \end{aligned} \tag{20}$$

Using the formulæ

$$\int_0^\infty d\lambda \int_q^p \lambda\rho\phi(\rho)\, \mathrm{J}_n(\lambda\rho)\, \mathrm{J}_n(\lambda r)\, d\rho = \phi(r), \qquad q < r < p,$$

$$\int_0^\infty \mathrm{J}_0(\lambda\rho)\, \mathrm{J}_1(\lambda a)\, d\lambda = \frac{1}{a}, \qquad \rho < a,$$

we find that (20) reduces to

$$\mathrm{U}_2 = -\int_0^l 2\pi a x \left(\frac{\partial \mathrm{V}_1}{\partial\rho}\right)_a dx - 4\pi a \int_0^a \left(\frac{\partial \mathrm{V}_1}{\partial x'}\right)_l \rho' d\rho' \int_0^\infty e^{-\lambda l} \sinh(\lambda l)\, \mathrm{J}_0(\lambda\rho')\, \mathrm{J}_1(\lambda a) \frac{d\lambda}{\lambda}$$

$$+ 4\pi a l \int_0^l \left(\frac{\partial \mathrm{V}_1}{\partial\rho'}\right)_a a\,dx' \int_0^\infty e^{-\lambda l} \cosh(\lambda x')\, \mathrm{J}_0(\lambda a)\, \mathrm{J}_1(\lambda a)\, d\lambda. \qquad (21)$$

We substitute now for V_1 from (10) and write

$$x = a\xi; \quad \rho' = a\eta; \quad l = am; \quad \mu = \lambda a; \quad \mu_1 = \lambda' a.$$

Hence we obtain

$$\mathrm{U}_2 = 2\pi \mathrm{H} a^3 \int_0^m \xi d\xi \int_0^\infty e^{-m\mu_1} \sinh(\mu_1\xi)\, \mathrm{J}_1^2(\mu_1)\, d\mu_1$$

$$-4\pi \mathrm{H} a^3 \int_0^1 \eta d\eta \int_0^\infty e^{-m\mu_1} \cosh(m\mu_1)\, \mathrm{J}_0(\mu\eta)\, \mathrm{J}_1(\mu_1)\, d\mu_1$$

$$\int_0^\infty e^{-m\mu} \sinh(m\mu)\, \mathrm{J}_0(\mu\eta)\, \mathrm{J}_1(\mu) \frac{d\mu}{\mu}$$

$$-4\pi \mathrm{H} a^2 l \int_0^m d\xi \int_0^\infty e^{-m\mu_1} \sinh(\mu_1\xi)\, \mathrm{J}_1^2(\mu_1)\, d\mu_1$$

$$\int_0^\infty e^{-m\mu} \cosh(\mu\xi)\, \mathrm{J}_0(\mu)\, \mathrm{J}_1(\mu)\, d\mu. \qquad (22)$$

The first two terms give the following:

$$2\pi \mathrm{H} a^3 m \int_0^\infty e^{-m\mu_1} \cosh(m\mu_1)\, \mathrm{J}_1^2(\mu_1) \frac{d\mu_1}{\mu_1} - 4\pi \mathrm{H} a^3 \int_0^\infty e^{-m\mu} \sinh(m\mu)\, \mathrm{J}_1^2(\mu) \frac{d\mu}{\mu^2}$$

$$-2\pi \mathrm{H} a^3 \int_0^1 \eta d\eta \int_0^\infty e^{-2m\mu_1} \mathrm{J}_0(\mu_1\eta)\, \mathrm{J}_1(\mu_1)\, d\mu_1 \int_0^\infty e^{-m\mu} \sinh(m\mu)\, \mathrm{J}_0(\mu\eta)\, \mathrm{J}_1(\mu) \frac{d\mu}{\mu}.$$

Also, in the third term of (22) we perform the integration with respect to ξ; then in all the terms of U_2 we apply the previous method of approximation, that is, we consider m large and substitute the lower terms of the expansions of the Bessel functions for small values of the argument. Using also the relation

$$\int_0^\infty \frac{\mathrm{J}_1^2(\mu)}{\mu}\, d\mu = \tfrac{1}{2},$$

and keeping to the order in which the parts of U_2 are given above, the approximation indicated gives, to the order $1/m^3$,

$$U_2 = \pi H a^2 l\left(\frac{1}{2}+\frac{1}{16m^2}-\frac{8}{3\pi m}+\frac{1}{4m^2}-\frac{1}{6\pi m^3}-\frac{1}{2}+\frac{1}{8\pi m^3}-\frac{1}{16m^2}\right)$$

$$= \pi H a^2 l\left(-\frac{8}{3\pi m}+\frac{1}{4m^2}-\frac{1}{24\pi m^3}\right). \qquad (23)$$

Comparing U_2 with U_1 given in (17), we see that they both begin with a term of the same order $1/m$; but we have

$$U_1+U_2 = -\frac{1}{24}\pi H a^2 l \frac{1}{m^3}. \qquad (24)$$

Returning now to the series (3) and (4) for the induced potential Q, we see that in calculating the total moment of the rod in this manner it is of advantage to use the series

$$Q = \frac{4\pi\kappa}{\mu}V_1+\left(\frac{4\pi\kappa}{\mu}\right)^2(V_1+V_2)+\ldots, \qquad \mu = 1+4\pi\kappa.$$

Then, if we denote by I_m the mean moment per unit volume obtained by calculating the moment of the whole bar, we have from (13), (17), and (24)—as far as the order indicated—

$$I_m = \kappa H\left\{1-\frac{4\pi\kappa}{\mu}\left(\frac{4}{3\pi m}-\frac{1}{8m^2}\right)+\left(\frac{4\pi\kappa}{\mu}\right)^2\frac{1}{48m^3}\right\}. \qquad (25)$$

§ 5. *The Mean Induction through the Central Section.*

Another quantity which we require to calculate is the mean polarisation over the central transverse section of the bar; if this be denoted by I_c, we have

$$I_c = -\frac{\kappa}{\pi a^2}\int_0^a 2\pi\rho\left(\frac{\partial\phi}{\partial x}\right)_{x=0}d\rho, \qquad (26)$$

where $\phi = -Hx+Q.$

To evaluate the contributions made by V_1 and V_2, we have from (10),

$$\frac{1}{\pi a^2}\int_0^a 2\pi\rho\left(\frac{\partial V_1}{\partial x}\right)_0 d\rho = \frac{2H}{a}\int_0^a \rho d\rho\int_0^\infty e^{-\lambda l}J_0(\lambda\rho)J_1(\lambda a)\,d\lambda$$

$$= 2H\int_0^\infty e^{-m\mu}J_1^2(\mu)\,d\mu/\mu$$

$$= H\left(\frac{1}{2m^2}-\frac{3}{4m^4}\right), \qquad (27)$$

to the order indicated, using previous method of approximation.

From the general expression in (9) we have V_2 given in the form

$$V_2 = -\mathrm{H}a\int_0^a \rho' d\rho' \int_0^\infty e^{-\lambda_1 l}\cosh(\lambda_1 l)\,\mathrm{J}_0(\lambda_1\rho')\,\mathrm{J}_1(\lambda_1 a)\,d\lambda_1 \int_0^\infty e^{-\lambda l}\sinh(\lambda x)\,\mathrm{J}_0(\lambda\rho)\,\mathrm{J}_0(\lambda\rho')\,d\lambda$$

$$+\mathrm{H}a^2\int_x^l dx' \int_0^\infty e^{-\lambda_1 l}\sinh(\lambda_1 x')\,\mathrm{J}_1{}^2(\lambda_1 a)\,d\lambda_1 \int_0^\infty e^{-\lambda x'}\sinh(\lambda x)\,\mathrm{J}_0(\lambda\rho)\,\mathrm{J}_0(\lambda a)\,d\lambda$$

$$+\mathrm{H}a^2\int_0^x dx' \int_0^\infty e^{-\lambda_1 l}\sinh(\lambda_1 x')\,\mathrm{J}_1{}^2(\lambda_1 a)\,d\lambda_1 \int_0^\infty e^{-\lambda x}\cosh(\lambda x')\,\mathrm{J}_0(\lambda\rho)\,\mathrm{J}_0(\lambda a)\,d\lambda. \qquad (28)$$

Hence we have

$$\left(\frac{\partial V_2}{\partial x}\right)_{x=0} = -\mathrm{H}a\int_0^a \rho' d\rho' \int_0^\infty e^{-\lambda_1 l}\cosh(\lambda_1 l)\,\mathrm{J}_0(\lambda_1\rho')\,\mathrm{J}_1(\lambda_1 a)\,d\lambda_1 \int_0^\infty \lambda e^{-\lambda l}\,\mathrm{J}_0(\lambda\rho)\,\mathrm{J}_0(\lambda\rho')\,d\lambda$$

$$+\mathrm{H}a^2\int_0^l dx' \int_0^\infty e^{-\lambda_1 l}\sinh(\lambda_1 x')\,\mathrm{J}_1{}^2(\lambda_1 a)\,d\lambda_1 \int_0^\infty \lambda e^{-\lambda x'}\,\mathrm{J}_0(\lambda\rho)\,\mathrm{J}_0(\lambda a)\,d\lambda. \qquad (29)$$

The mean value of this taken over the circle $x = 0$, $\rho = a$, is given by

$$\frac{1}{\pi a^2}\int_0^a 2\pi\rho\left(\frac{\partial V_2}{\partial x}\right)_0 d\rho$$

$$= -2\mathrm{H}\int_0^a \rho' d\rho' \int_0^\infty e^{-\lambda_1 l}\cosh(\lambda_1 l)\,\mathrm{J}_0(\lambda_1\rho')\,\mathrm{J}_1(\lambda_1 a)\,d\lambda_1 \int_0^\infty e^{-\lambda l}\mathrm{J}_1(\lambda a)\,\mathrm{J}_0(\lambda\rho')\,d\lambda$$

$$+2\mathrm{H}a\int_0^l dx' \int_0^\infty e^{-\lambda_1 l}\sinh(\lambda_1 x')\,\mathrm{J}_1{}^2(\lambda_1 a)\,d\lambda_1 \int_0^\infty e^{-\lambda x'}\,\mathrm{J}_0(\lambda a)\,\mathrm{J}_1(\lambda a)\,d\lambda. \qquad (30)$$

We proceed to evaluate these integrals for m large by substituting the expansions of the Bessel functions as before; then the first integral in (30) becomes, to the required order,

$$-\frac{\mathrm{H}}{a^2}\int_0^a \rho'\left\{1+\frac{1}{8m^2}-\frac{3}{16}\left(\frac{\rho'^2}{4a^2}+\frac{1}{8}\right)\frac{1}{m^4}\right\}\left\{\frac{1}{2m^2}-3\left(\frac{\rho'^2}{4a^2}+\frac{1}{8}\right)\frac{1}{m^4}\right\}d\rho'$$

$$= -\mathrm{H}\left(\frac{1}{4m^2}-\frac{13}{32}\frac{1}{m^4}\right). \qquad (31)$$

The second integral in (30) is equivalent to

$$\mathrm{H}\int_0^\infty\int_0^\infty\left\{\frac{e^{-m\mu}}{\mu_1-\mu}+\frac{e^{-m(\mu+2\mu_1)}}{\mu_1+\mu}-\frac{e^{-m\mu_1}}{\mu_1-\mu}-\frac{e^{-m\mu_1}}{\mu_1+\mu}\right\}\mathrm{J}_1{}^2(\mu_1)\,\mathrm{J}_0(\mu)\,\mathrm{J}_1(\mu)\,d\mu\,d\mu_1.$$

Substituting values of the Bessel functions of μ or μ_1, or of both, according to the exponential term, this reduces to

$$H\left(\frac{1}{4m^2}-\frac{35}{32}\frac{1}{m^4}\right). \tag{32}$$

Adding this to the part given in (31), we see that the mean value of $\partial V_2/\partial x$ over the central section begins with a term $-23H/32m^4$, while the mean value of $\partial V_1/\partial x$ begins with a term of the order $1/m^2$.

Then, substituting in (26) the values calculated above, we have, as far as the order shown,

$$\begin{aligned} I_c &= \kappa H-\kappa\left\{4\pi\kappa H\left(\frac{1}{2m^2}-\frac{3}{4m^4}\right)-(4\pi\kappa)^2 H\frac{23}{32m^4}\right\} \\ &= \kappa H\left\{1-\frac{2\pi\kappa}{m^2}+\pi\kappa\left(3+\frac{23}{2}\pi\kappa\right)\frac{1}{m^4}\right\}. \end{aligned} \tag{33}$$

§ 6. *Application to Dielectric Polarisation.*

If an ellipsoid of specific inductive capacity K is placed in a uniform electric field, it is known that the field within the ellipsoid is uniform, and the polarisation at any point is given by a formula

$$I = \kappa(H-NI); \qquad \kappa = (K-1)/4\pi. \tag{34}$$

The factor N is a numerical factor independent of I, and can be calculated theoretically; for a prolate spheroid, of which m is the ratio of the longer to the shorter axis, we have

$$N = \frac{4\pi}{m^2-1}\left\{\frac{m}{\sqrt{m^2-1}}\log_e(m+\sqrt{m^2-1})-1\right\}, \tag{35}$$

the axis of revolution being along the direction of the field. Also, for an infinite rod of circular section, N is zero for a uniform field parallel to the axis of the cylinder, and is equal to 2π for a field at right angles to the axis.

In the case of a rod of finite length placed in a uniform field, there is no longer uniformity of the field within the rod, but as we suppose the specific inductive capacity to be independent of the field, we have still an equation of the form (34). For a field parallel to the axis of the rod, we have from (25) to the first order of $1/m$, the ratio of diameter to length,

$$N = 16/3Km. \tag{36}$$

For a field at right angles to the axis we may still take N to be equal to 2π. Now, in determinations of the specific inductive capacity of a dielectric in the form of a cylindrical rod, it is necessary to measure the couple required to keep the rod at a given angle ϕ to the direction of the external field. If we put V for the volume of the rod and use the values of N obtained above, we have

$$\text{Couple on rod} = \kappa^2 \mathrm{V} \frac{2\pi - 16/3\mathrm{K}m}{(1+2\pi\kappa)(1+16\kappa/3\mathrm{K}m)} \mathrm{H}^2 \sin\phi\cos\phi, \tag{37}$$

where

$$\kappa = (\mathrm{K}-1)/4\pi.$$

§ 7. *The Demagnetising Factor of a Finite Rod.*

When the problem is one of magnetic induction, it is clear that the previous results only supply an approximation to a solution; for, since the field is not uniform within the rod, it follows that the magnetic permeability μ varies along the rod, thus complicating the problem to be solved. In fact, the demagnetising force will not be expressible in the form NI, where N is a numerical factor independent of I.

On the supposition that there is such a constant N, at least approximately, for long thin rods, Du Bois* made a comparison between the values of N for ovoids and cylindrical rods of the same dimension-ratio m; the factors for the rods were calculated from experiments by Ewing for the longer rods and by Tanakadaté for the shorter ones.

It was pointed out by Mann† that in the former experiments ballistic methods were used, but in the latter magnetometric methods. Consequently the quantities measured were those denoted by I_m and I_c above; and the factor N deduced by the two methods will, in general, be different, and may be denoted by N_m and N_b, being the magnetometric and ballistic values of the demagnetisation factor. From the series given in (25) and (33) we have, for large values of m, the approximate values

$$\mathrm{N}_m = \frac{16}{3\mu m}; \qquad \mathrm{N}_b = \frac{2\pi}{m^2}. \tag{38}$$

Benedicks‡ further investigated the values of N_m and N_b for rods and, in particular, their variation with the intensity of magnetisation; the value of the demagnetising force $\mathrm{N}_b\mathrm{I}$ at the centre of the rod was found to increase with the field, and then after a certain point to decrease rapidly.

Using the ballistic method, Lamb§ determined the induction through sections taken at intervals along a bar, and was thus able to trace curves showing the distribution of induction and force along the bar and the consequent effective distribution of magnetic permeability.

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* Du Bois, 'Annalen der Physik,' vol. 46, p. 495, 1892.

† C. R. Mann, 'Physical Review,' vol. 3, p. 359, 1895.

‡ Benedicks, 'Annalen der Physik,' vol. 6, p. 726, 1901.

§ C. G. Lamb, 'Phil. Mag.,' vol. 48, p. 262, 1899.

Searle and Bedford* investigated the demagnetising force h at the centre of a long cylindrical rod. Determining the values of h for increasing values of the external force H, they found that h increases at first more rapidly than $2\pi I_c/m^2$, but after passing a maximum it descends to a minimum somewhat less than that value; the latter result is interpreted as being due to hysteresis effects.

A quantity measured in several experiments was the ratio I_m/I_c, giving the effective length of the bar.

Using expressions (25) and (33) above, we have approximately

$$I_m/I_c = 1-\left(\frac{4\pi\kappa}{\mu}\right)\frac{4}{3\pi m}.$$

With a rod for which m was 25, Benedicks found this ratio increased to 0·84 at the highest value of H used in the experiment.

Reviewing the experimental results, it appears that in the case of magnetic induction the facts of the non-uniformity of the force within the rod and the consequent variation of effective susceptibility are too important to be neglected; hence, in this case, the calculations of the previous sections supply limiting values towards which the quantities approach as the magnetisation tends to saturation.

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* Searle and Bedford, 'Phil. Trans.,' A, vol. 198, p. 98, 1902.